Advances and Applications in Discrete Mathematics

(ISSN 0974-1658)

Aims and Scope: The Advances and Applications in Discrete Mathematics is an international journal, which publishes high quality original research papers as well as state-of-the-art survey articles related to all aspects (theory and applications) of different parts of *Discrete Mathematics*.

Indexing and Reviews: Mathematical Reviews, MathSciNet, Zentralblatt für Mathematik.

Paper Submission: Authors may submit their papers for consideration in the Advances and Applications in Discrete Mathematics by the following modes:

A. **Online submission (only .pdf and .doc files):** Please visit journal's homepage:
http://pphmj.com/journals/aadm.htm

B. **Electronically (only .pdf, .tex, .dvi, .ps and .doc files):** At the e-mail address:
aadm@pphmj.com or arun@pphmj.com

C. **Hard copies:** Papers in duplicate with a letter of submission at the following address:
The Editors, Advances and Applications in Discrete Mathematics
Pushpa Publishing House, Vijaya Niwas, 198, Mumfordganj
Allahabad 211002, India, Tel.: 0091-532-2250078

The paper must be typed only on one side in double spacing with a generous margin all round. An effort is made to publish a paper duly recommended by a referee within a period of six months. One set of galley proofs will be sent to the author submitting the paper, unless requested otherwise, without the original manuscript, for corrections.

Authors are requested to provide an abstract of not more than 200 words and the latest Mathematics Subject Classification. Statements of Lemmas, Propositions and Theorems should be set in *italics* and references should be arranged in alphabetical order by the surname of the first author.

Print Charges and Reprints: Authors are requested to arrange print charges of their papers @ USD 40 per page for USA and Canada, and EURO 30 per page for rest of the world from their institutions/research grants, if any. However, for authors in India this charge is Rs. 500 per page. Twenty five reprints of a paper are provided to the authors ex-gratis. Additional copies of reprints may be ordered at the time of proof correction.

2008

ISSN 0974-1658

Advances and Applications in Discrete Mathematics

Reprint

Pushpa Publishing House
Vijaya Niwas, 198, Mumfordganj
Allahabad 211002, India
Tel.: 0091-532-2250078
Fax: 0091-532-2641508
e-mail: arun@pphmj.com

Advances and Applications in Discrete Mathematics
Volume 1, Number 2, 2008, Pages 187-194
This paper is available online at http://www.pphmj.com

ALGEBRAIC PRESENTATION OF CLASSICAL NETS

AHMAD N. AL-KENANI

Department of Mathematics
King Abdulaziz University
P. O. Box 80219
Jeddah 21589, Saudi Arabia
e-mail: aalkenani10@hotmail.com

Abstract

The geometric construction of a (μ, m)-net is known, see for example [2] and [4]. In this paper, we discuss the algebraic presentation of a very important type of symmetric nets, which is called classical symmetric nets, and begin by giving the algebraic presentation of the classical $(1, q)$-nets, where q is a prime power and then provide its generalization to get the algebraic presentation of classical symmetric (q^{n-2}, q)-nets, where q is a prime power and $n \geq 2$ is integer. Finally, we give the generalization of this construction.

Introduction

A $t-(v, k, \lambda)$ design $\mathcal{D}$ is an incidence structure with v points, k points on a block and any subset of t points is contained in exactly λ blocks, where $v > k$, $\lambda > 0$. The number of blocks is denoted by b and the number of blocks on a point by r. $\mathcal{D}$ is symmetric if $b = v$ or, equivalently, $r = k$. $\mathcal{D}$ is resolvable if its blocks can be partitioned into subsets, of m blocks, called *parallel classes*, such that each class partitions the point set of $\mathcal{D}$. In this case, two blocks are said to be

2000 Mathematics Subject Classification: Primary 05B25; Secondary 51E14, 51E15.

Keywords and phrases: affine and projective planes, affine design, classical net.

This research was supported by Deanship of Scientific Research at King Abdulaziz University, Jeddah, Saudi Arabia, 2007.

Received February 19, 2008

parallel if they are in the same parallel class and *non-parallel* otherwise. The number of parallel classes is r.

$\mathcal{D}$ is called *affine* if it is resolvable so that any two non-parallel blocks meet in a constant number μ of points. It is easy to show that $m = v/k$ and, if $r > 1$, then $\mu = k/m$. See for example [2], or [3] for more details. The dual $\mathcal{D}^*$ of a design $\mathcal{D}$ is the incidence structure whose points and blocks are, respectively, the blocks and points of $\mathcal{D}$ with induced incidence. Affine 1–designs are also called *nets*, see [2]. Affine 1–designs $\mathcal{D}$ for which $\mathcal{D}^*$ is also affine are necessarily symmetric and are called *symmetric nets*. In this case $b = v = \mu m^2$ and $k = r = \mu m$. That is, $\mathcal{D}$ is an affine $1 - (\mu m^2, \mu m, \mu m)$ design whose dual $\mathcal{D}^*$ is also affine with the same parameters. For short we call such a symmetric net a (μ, m)*-net*. The parameters μ, m are, respectively, the index and class number of the net.

If $\mathcal{D}$ is a symmetric net we shall refer to the parallel classes of $\mathcal{D}$ as block classes of $\mathcal{D}$ and to the parallel classes of $\mathcal{D}^*$ as point classes of $\mathcal{D}$.

Let $\mathcal{D}$ be an affine $1 - (v, k, r)$ design whose dual $\mathcal{D}^*$ is resolvable. Then $\mathcal{D}$ is symmetric if and only if $\mathcal{D}^*$ is affine (see [5]).

Let q be a prime power and let F be the field $GF(q)$. The points and hyperplanes of the projective geometry $PG(n, q)$ are, respectively, the 1–dimensional and the n–dimensional subspaces of the $(n+1)$–dimensional vector space $V_{n+1}(q)$ over F.

A point P of the projective geometry can be represented in *homogeneous* coordinates $\mathbf{x} = (x_0, x_1, \ldots, x_n)$, where not all the x_i are 0 and if $0 \neq \lambda \in F$, the P is also represented by $\lambda\mathbf{x}$.

Similarly, a hyperplane A can be represented by homogeneous coordinates $\mathbf{a}'$. The point P is on the hyperplane A if and only if

$$\mathbf{x}\mathbf{a}' = 0 = \sum_{i=0}^{n} x_i a_i.$$

The points and hyperplanes of $PG(n, q)$ form a symmetric

$$2 - \left(\frac{q^{n+1} - 1}{q - 1}, \frac{q^n - 1}{q - 1}, \frac{q^{n-1} - 1}{q - 1}\right)$$

design. Let U be any hyperplane in the projective geometry $PG(n, q)$ and let u be any point on U. Deleting U from $PG(n, q)$ gives the affine geometry $AG(n, q)$. The points and hyperplanes of $AG(n, q)$ form an affine $2 - \left(q^n, q^{n-1}, \frac{q^{n-1} - 1}{q - 1}\right)$ design. Also delete all hyperplanes of $PG(n, q)$ that lie on u. The remaining points and hyperplanes form the classical symmetric net (also known as the Desarguesian symmetric net) with $m = q$ and $\mu = q^{n-2}$, where q is a prime power.

Algebraic Presentation of $(1, q)$-net

First we consider the special situation of the classical or Desarguesian $(1, q)$-nets, where q is a prime power and then describe the more general case.

The classical $(1, q)$-net D is obtained by deleting a parallel class of lines from the unique (up to isomorphism) affine plane of order q (over the field $F = GF(q)$).

Let $\mathcal{D}$ be $(1, q)$-net. Then the algebraic presentation of $\mathcal{D}$ can be given as follows:

Points: Ordered triples $(x, y, 1)$, $x, y \in GF(q)$.

Lines: Ordered triples $[1, p, r]$, $p, r \in GF(q)$.

Incidence: $(x, y, 1) I [1, p, r] \Leftrightarrow x + py + r = 0$.

This net is a self-dual net.

We give an outline of the proof of this.

Consider the map $\psi : D \to D^*$ defined as follows:

$\psi : (x, y, 1) \to [1, y, x]$ on points.

$[1, p, r] \to (r, p, 1)$ on blocks.

Clearly ψ is bijective. Now we show ψ is an isomorphism:

$$\begin{aligned} \psi : (x, y, 1)I[1, p, r] &\Leftrightarrow x + py + r = 0 \\ &\Leftrightarrow r + py + x = 0 \\ &\Leftrightarrow (r, p, 1)I[1, y, x] \\ &\Leftrightarrow \psi(x, y, 1)I\psi[1, p, r]. \end{aligned}$$

This proves that ψ is an isomorphism.

Hence D is isomorphic to D^*.

A parallel class of points or of lines is determined uniquely by an element of $GF(q)$. This is easily verified.

For any $c \in GF(q)$, the points $(x, c, 1)$, $x \in GF(q)$, form a point class and for any $e \in GF(q)$, the lines $[1, e, r]$, $r \in GF(q)$, form a line class.

So we can represent a point class or line class uniquely by an element of $GF(q)$.

Algebraic Representation of (q^{n-2}, q)-net

Now we shall give the algebraic representation of (q^{n-2}, q)-net.

Let $F = GF(q)$. Define a design $\prod$ as follows:

Let $n \geq 2$ be an integer. Then

Points: Ordered n-tuple $(x_1, x_2, ..., x_n)$, where $x_i \in F, 1 \leq i \leq n$.

Blocks: Given $a_1, a_2, ..., a_{n-1}$, $b \in F$ and the subset of points $(x_1, x_2, ..., x_n)$ that satisfy the equation

$$a_1x_1 + a_2x_2 + \cdots + a_{n-1}x_{n-1} + x_n = b$$

is a block of $\prod$, denoted by the ordered n-tuple $[a_1, a_2, a_3, ..., a_{n-1}; b]$.

Proof. Clearly $\prod$ has q^n points and q^n blocks. The number of points on a block $[a_1, a_2, a_3, ..., a_{n-1}; b]$ is q^{n-1}. To see this, note that we can choose x_i, $i = 1, 2, ..., n-1$ arbitrarily (this can be done in q^{n-1} ways) and then x_n is determined uniquely by the equation of the block.

We show that $\prod$ is resolvable as follows. Given $a_1, a_2, ..., a_{n-1} \in F$, the set of blocks $[a_1, a_2, ..., a_{n-1}; b]$, where $b \in F$ forms a parallel class of blocks. To see this, note that given any point $p = (p_1, p_2, ..., p_n)$, then p is on the unique block of this set with $b = \sum_{i=1}^{n} a_ip_i + p_n$.

Therefore, these blocks partition the points of $\prod$.

Thus $\prod$ is resolvable since its blocks can be partitioned into parallel classes.

Now we show $\prod$ is affine. Consider two non-parallel blocks with equations:

$$a_1x_1 + a_2x_2 + \cdots + a_{n-1}x_{n-1} + x_n = b, \quad (1)$$

$$a_1'x_1 + a_2'x_2 + \cdots + a_{n-1}'x_{n-1} + x_n = b'. \quad (2)$$

Since the blocks are not parallel, $a_i \neq a_i'$, for all i. Without loss of generality, we may assume $a_1 \neq a_1'$.

Now

$$(1) - (2) \Rightarrow (a_1 - a_1')x_1 + (a_2 - a_2')x_2 + \cdots + (a_{n-1} - a_{n-1}')x_{n-1} = b - b'. \quad (3)$$

Since $a_1 - a_1' \neq 0$, given any values in F for $x_2, x_3, ..., x_{n-1}$ (q^{n-2} choices), we can find x_1 uniquely to satisfy (3). Then x_{n-1} follows uniquely from (1) or (2).

Therefore, blocks (1) and (2) meet in exactly q^{n-2} points.

Hence $\prod$ is affine.

To show that the dual of $\prod$ $(\prod^*)$ is affine too, according to Hine and Mavron theorem (see [5]), it is enough to show $\prod^*$ is resolvable.

It is easy to verify that given $x_1, x_2, \ldots, x_{n-1} \in F$, the q points $(x_1, x_2, \ldots, x_{n-1}, y)$, $y \in F$ form a point parallel class in Π. A given block $[a_1, a_2, \ldots, a_{n-1}; b]$ is on the unique point of this point class with

$$y = b - \sum_{i=0}^{n-1} a_i x_i.$$

Alternatively, the fact that Π^* is affine follows by showing that in fact Π is self-dual. It is easily checked that the following correspondence between points and blocks of Π is a polarity (i.e., an isomorphism map whose square is the identity):

$$(x_0, x_1, \ldots, x_n) \leftrightarrow [x_0, x_1, \ldots, x_{n-1}, -x_n]$$

to see that the algebraic representation we gave previously is isomorphic to the classical net, with U the hyperplane with homogeneous coordinates $[1, 0, 0, \ldots, 0]$ and u is the point with homogeneous coordinates $(0, 0, \ldots, 0, 1)$.

First note that each point not on U has *unique* homogeneous coordinates of the form $(1, x_1, \ldots, x_n)$, since the first coordinate is not 0. Similarly, a hyperplane not on u has *unique* homogeneous coordinates of the form $(a_0, a_1, \ldots, a_{n-1}, 1)$. The correspondence given below between points and blocks of Π and the points and hyperplanes, respectively, of the hyperplane obtained by deleting from $PG(n, q)$ the hyperplane U and the points on U and dually for u,

$$(x_1, x_2, \ldots, x_n) \leftrightarrow (1, x_1, x_2, \ldots, x_n)$$

$$[a_1, a_2, \ldots, a_{n-1}; b] \leftrightarrow [-b, a_1, a_2, \ldots, a_{n-1}, 1]$$

is obviously bijective. We can readily see that it is incidence preserving and therefore an isomorphism, as follows:

$$(x_1, x_2, \ldots, x_n) I [a_1, a_2, \ldots, a_{n-1}; b] \text{ in the net}$$

$$\Leftrightarrow a_1x_1 + a_2x_2 + \cdots + a_{n-1}x_{n-1} + x_n = b$$

$$\Leftrightarrow -b + a_1x_1 + a_2x_2 + \cdots + a_{n-1}x_{n-1} + x_n = 0$$

$$\Leftrightarrow (1x_1x_2 \cdots x_n) \text{ is on } [-ba_1a_2 \cdots a_{n-1}, 1] \text{ in } PG(n, q).$$

Generalizations

The construction of the classical symmetric (q^{n-2}, q)-nets can be generalized by replacing the field $GF(q)$ with more general algebraic structure F. For example, F could be a right nearfield. This is essentially like a field but without the requirement that the left distributive law $x(y + z) = xy + xz$ holds. A left nearfield is defined analogously. A proper nearfield is one which is not a field. The smallest order of a proper nearfield is 9.

If $(F, +, \cdot)$ is a right nearfield, then $F^* = (F, +, \odot)$ is a left nearfield, where $x \cdot y = y \odot x$, for all $x, y \in F$.

Another example is when F is a semifield. A semifield satisfies all conditions for a field except that under multiplication the non-zero elements form a loop but not necessarily a group.

Characterization of the Classical Symmetric Nets

The classical (q^{n-2}, q)-symmetric nets can be characterized combinatorially for the case $n > 2$. The *line* joining two non-parallel points in a net is the intersection of all blocks containing the two points. In a (μ, m)-net N, a line has most m points. Furthermore (see Mavron [10]), if $\mu > 1$, then N is a classical symmetric net if and only if every line has m points.

The case $\mu = 1$ is different. In this case, a symmetric $(1, m)$-net is just an affine plane of order m with the lines of one parallel class deleted. For any affine plane, Desarguesian or not, the resulting structure will always satisfy the condition that all lines have m points.

Acknowledgement

The author would like to thank Professor V. C. Mavron for reading carefully the manuscript and suggesting several improvements.

References

[1] A. N. Al-Kenani and V. C. Mavron, Non-tactical symmetric nets, J. London Math. Soc. (2) 67 (2003), 273-288.

[2] T. Beth, D. Jungnickel and H. Lenz, Design Theory, Cambridge University Press, Cambridge, 1999.

[3] C. J. Colbourn and J. H. Dinitz, The CRC Handbook of Combinatorial Designs, CRC Press, Boca Raton, New York, London, Tokyo, 1996.

[4] P. Dembowski, Finite Geometries, Springer-Verlag, New York, 1968.

[5] T. Hine and V. C. Mavron, Embeddable transversal designs, Discrete Math. 29 (1980), 191-200.

[6] Y. J. Ionin and M. S. Shrikhande, Combinatorics of Symmetric Designs, Cambridge University Press, Cambridge, 2006.

[7] D. Jungnickel, On difference matrices, resolvable transversal designs and generalised Hadamard matrices, Math. Z. 167 (1979), 49-60.

[8] V. C. Mavron, Resolvable designs, affine designs and hypernets, Preprint No. 99, FPS Kombinatorische Mathematik, Freie University, Berlin, 1979.

[9] V. C. Mavron, Constructions for resolvable and related designs, Aequationes Math. 23 (1981), 131-145.

[10] V. C. Mavron, A characterisation of some symmetric substructures of projective and affine geometries, Arch. Math. 36 (1981), 281-288.

■